HSC
Math Ext. 1.

Polynomials I.

INTRODUCTION TO POLYNOMIALS

DEFINITION AND TERMINOLOGY

A polynomial is an algebraic expression of the form:

Definition of a Polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0 = \sum_{i=0}^{n} a_i x^i$$

The above expression is known as the polynomial $P(x)$.

- Polynomials are continuous and differentiable at every point.
- The exponents of a polynomial are non-negative integers. Hence, equations containing $5x^{\frac{3}{2}}$ or $3x^{-4}$ are not polynomials.
- Coefficients are real numbers (at least for the scope of the extension 1 HSC Course)

Term	**Symbol**	**Definition**	**Significance**
Degree	n	Highest power of x	Determines the general shape of the curve
Coefficient	$a_1, a_2 \dots, a_n$	Constants in front of powers of x	
Leading Coefficient	a_n	Constant in front of highest power of x	Determines the behavior of the graph for large values of x.
Constant Term	a_0	Constant term that does not change even as x varies	Constant term, and is the y intercept
Zeroes	$\alpha_1, \alpha_2, \dots \alpha_n$	Values of x for which $P(x) = 0$	x-intercepts of the Polynomial. A polynomial of degree n may have *up to* n roots

TYPES OF POLYNOMIALS

Some polynomials have special names:

Degree (n)	**Polynomial $P(x)$**	**Special Name**
0	$P(x) = a_0$	Constant Polynomial
1	$P(x) = a_1x + a_0$	Linear Polynomial
2	$P(x) = a_2x^2 + a_1x + a_0$	Quadratic Polynomial
3	$P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$	Cubic Polynomial
4	$P(x) = a_4x^4 + \ a_3x^3 + a_2x^2 + a_1x + a_0$	Quartic Polynomial
n	$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$	Monic Polynomial

ROOTS & ZEROES

The equation $P(x) = 0$ is known as a polynomial equation of degree n where n is the degree of the polynomial P. The real number c such that $P(c) = 0$ is known as a root (or solution) of the polynomial equation, as well as a zero of the polynomial P. A polynomial equation may have more than one root (up to n roots, where n is the degree of the polynomial) or may have none at all (e.g. $x^2 + 1 = 0$)

A polynomial has 'zeroes' and a polynomial equation has 'roots'.

Question 1

Identify which of these following expressions are polynomials. For the polynomials you have identified, identify the degree, leading coefficient and constant.

a) $2x^3 + 24x - 15$

b) $x^6 + 3x^3 - 2x^{-2}$

c) $3x^2 - 4x + x^{\frac{5}{4}}$

d) $2(x+1)(x-4)(x+3)$

e) $x^5 + 5^x$

f) 0

Question 2

Consider the polynomial $P(x) = (x - 1)(x + 1)(x + 2)$

a) Write down the zeroes of $P(x)$

b) By expanding $P(x)$, find the roots of the polynomial equation $x^3 + 2x^2 - x - 2 = 0$

OPERATIONS ON POLYNOMIALS

Performing operations with polynomials is a relatively straightforward exercise. In order to perform addition or subtraction, we group like terms and add or subtract coefficients.

Multiplication of two polynomials is performed according to the normal expansion method.

Degree of Sum and Difference

Given $P(x)$ and $Q(x)$ with degree *m* and *n* respectively then:

- If $n \neq m$, then $\deg\big(P(x) \pm Q(x)\big) = \max\,(m, n)$
- If $n = m$, then $\deg\big(P(x) \pm Q(x)\big) \leq n$

Division however is more complicated and will be looked at in detail in another section.

Degree of Product

Given non-zero polynomials $P(x)$ and $Q(x)$ with degree *m* and *n* respectively, then:

$$\deg\big(P(x) \cdot Q(x)\big) = m + n$$

Question 3

Given that $P(x) = x^2 + 2x - 1$ and $Q(x) = x^3 + x^2 + 1$, state the degree of the polynomial $R(x)$ if

a) $R(x) = P(x) + Q(x)$

b) $R(x) = P(x) \times Q(x)$

Question 4

Given that $P(x) = 2x^2 + x$ and $Q(x) = x - 1$, find

a) $P(x) - Q(x)$

b) $P(x) \times Q(x)$

Question 5

Consider the polynomials $P(x) = 2x^2 + 3x + 1$ and $Q(x) = -2x^2 + 1$

a) Find the degree of $P(x) + Q(x)$

b) State the degree of $P(x) \times Q(x)$

c) Verify the answer to b) via direct expansion of $P(x) \times Q(x)$

Question 6

Given $P(x) = x^3 + 3x$ and $Q(x) = 3x + 1$, find the degree of the following polynomials

a) $P(x) \times Q(x)$

b) $[P(x)]^2$

c) $P(x)[Q(x)]^2$

d) $[P(x)]^2 + P(x)[Q(x)]^2$

Graphing Polynomials

General Shape of the Curve

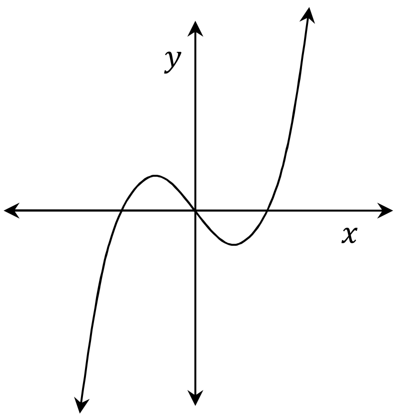
General Shape of a Polynomial

The general shape of a polynomial will be determined by whether the degree of the polynomial is even or odd.

Polynomials of Odd Degree

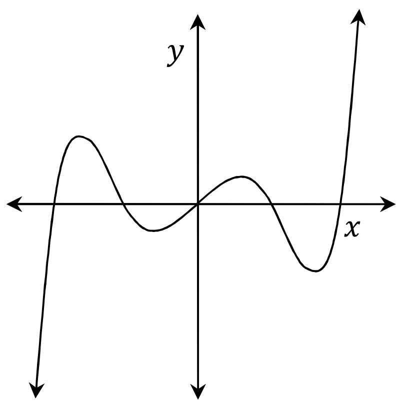
Question 7 (Conceptual)

The graph of two polynomials of odd degree are shown below:



$y = x^3 - x$

$= x(x-1)(x$



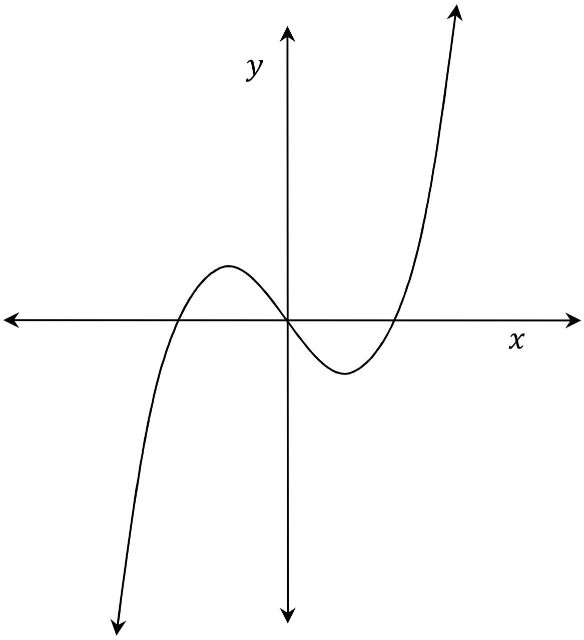
$y = x^5 - 5x^3 + 4x$

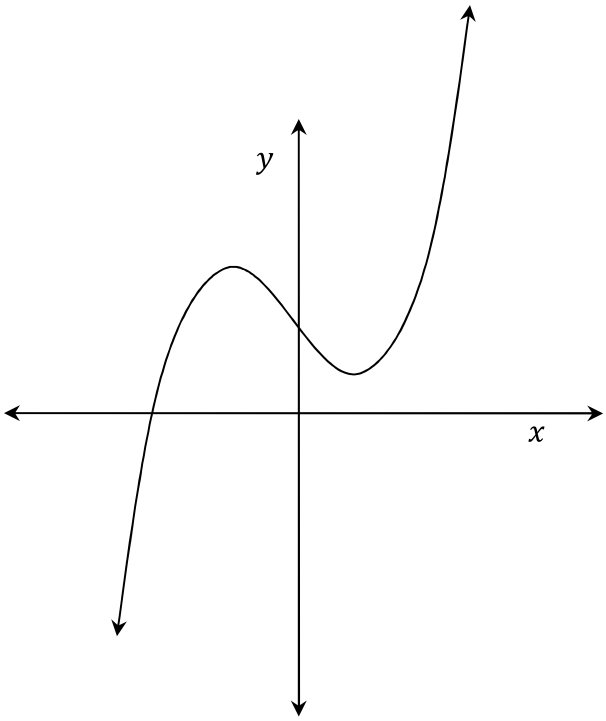
$= x(x-1)(x+1)(x-2)(x+2)$

a) What do you notice about the general shapes of the graphs?

b) What do you notice about the shape of the graph as $x \to \pm\infty$ (i.e. the tails of the curve)?

c) By examining the graphs of the cubic below, determine the minimum and maximum number of roots a cubic can have?



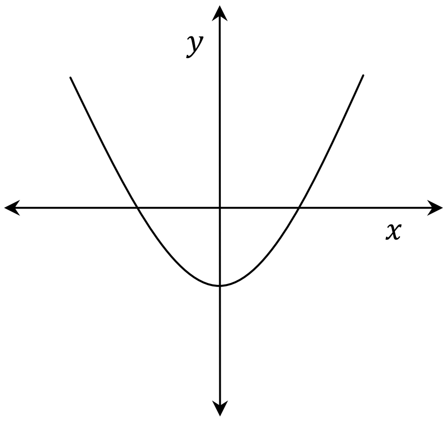


d) What is the minimum number of roots a polynomial with odd degree can have?

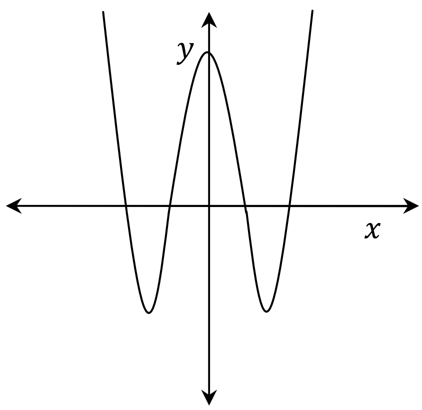
Polynomials of Even Degree

Question 8 (Conceptual)

The graphs of a quadratic and quartic polynomial are shown below:



$y = x^2 - 1$

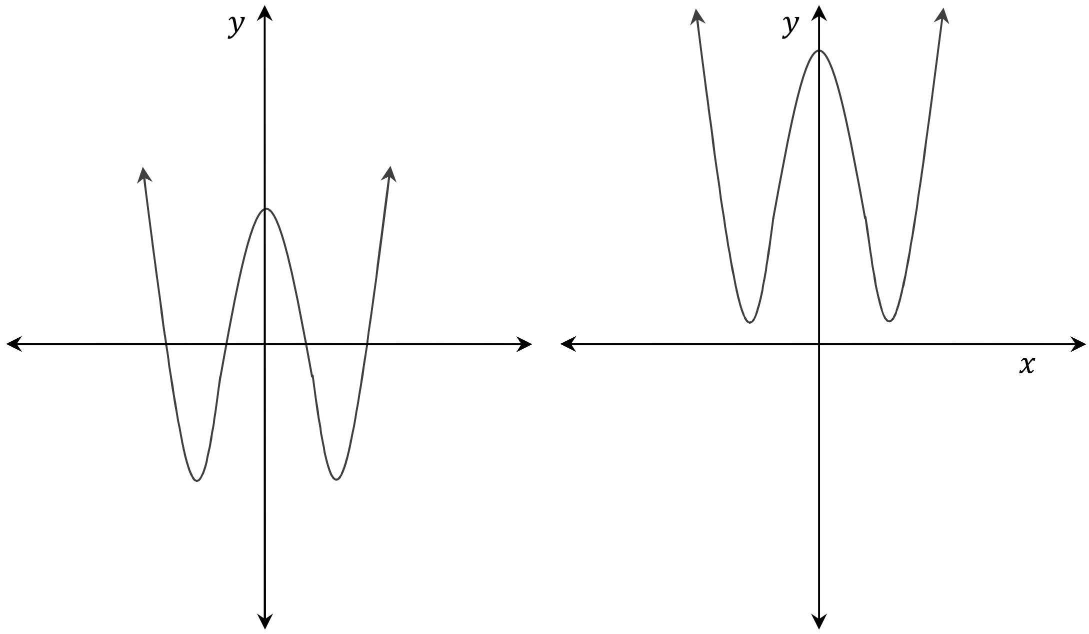


$y = x^4 - 5x^2 + 4$

a) What do you notice about the general shapes of the graphs?

b) What do you notice about the shape of the graph as $x \to \pm\infty$ (i.e. the tails of the curve)?

c) By examining the graph below, determine the minimum and maximum number of roots a quartic can have



d) What is the minimum number of roots a polynomial of even degree can have?

SUMMARY OF ODD AND EVEN DEGREE FUNCTIONS

Degree	**Direction of the Tails**	**Maximum Number of Roots (for Polynomial of degree n)**	**Minimum Number of Roots**
Odd			
Even			

BEHAVIOUR FOR LARGE $|x|$

For large values of $|x|$, the leading term a_nx^n dominates the function

We have established that x^n will affect the general shape of the curve (odd or even n). The sign of the leading coefficient determines the value as $x \to \pm\infty$

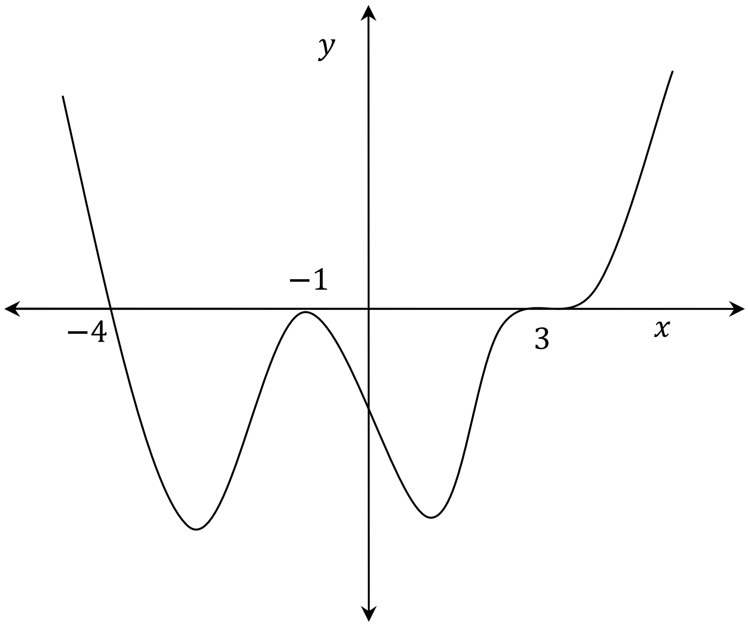
	$\boldsymbol{a_n} > 0$	$\boldsymbol{a_n} < 0$
$x \to \infty, n$ even or odd	$P(x) \to +\infty$	$P(x) \to -\infty$
$x \to -\infty, n$ even	$P(x) \to +\infty$	$P(x) \to -\infty$
$x \to -\infty, n$ odd	$P(x) \to -\infty$	$P(x) \to +\infty$

Talent Tip:

SHAPE OF SINGLE, DOUBLE AND TRIPLE ROOTS

Question 9 (Conceptual)

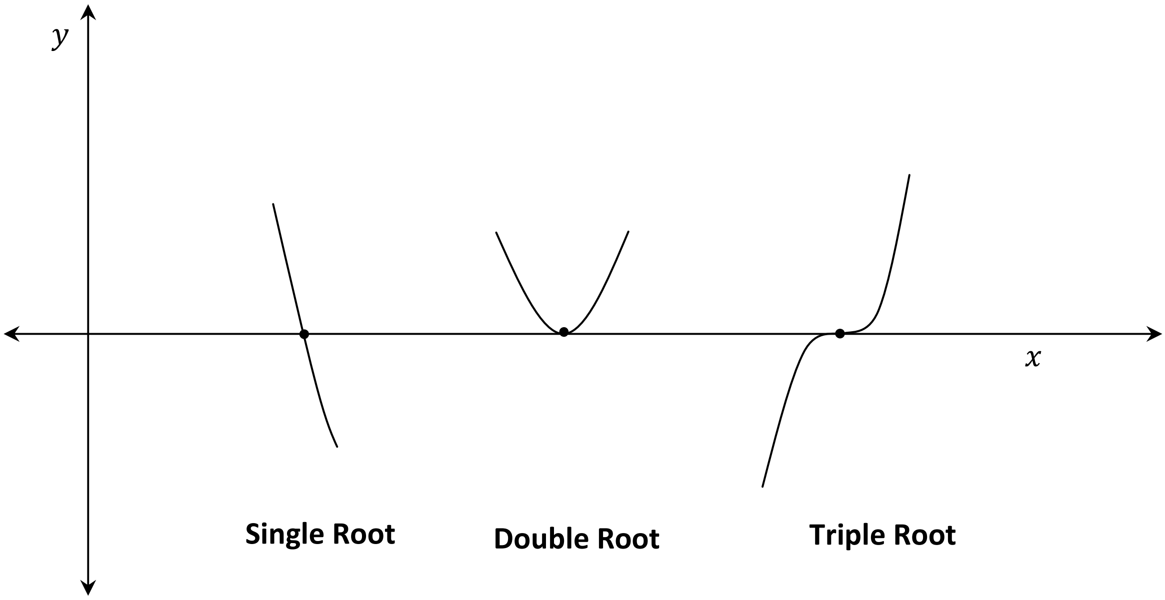
The graph below shows the curve $y = (x+4)(x+1)^2(x-3)^3$



a) What are the zeroes of the polynomial?

b) Compare the shape of the curve at the zeroes

The general shape of single, double and triple roots are seen below



Behaviour of Multiple Zeroes

- At a single zero, the curve cuts the *x*-axis, not tangent to it.
- At a zero of even multiplicity, the curve lies tangent to the *x*-axis without crossing it
- At a zero of odd multiplicity (≥ 3), the curve lies tangent to the *x*-axis and possesses a point of inflection at this root, crossing over the *x*-axis.

CURVE SKETCHING METHOD

A basic method for sketching polynomial functions:

STEP 1: Determine the general shape of the graph, and its behavior at the extremities by examining the leading term $a_n x^n$.

- When n is even, the tails of the curve will be on the same side of the $y - axis$ (like a quadratic)
- When n is odd, the tails of the curve will be on different sides (like a cubic)
- The sign of a_n determines the behavior at the extremeties.

Mark the "tails" of the polynomial

STEP 2: Determine the zeroes of the polynomial

STEP 3: Determine the nature of the zeroes, i.e. their multiplicity and hence their shape

STEP 4: Determine the *y*-intercept by evaluating $P(0)$.

STEP 5: Sketch the curve using the above information

Talent Tip:

Question 10

Graph the curve $y = (x^2 - 4)(x - 4)$

STEP 1: Determine the general shape of the graph

STEP 2: Determine the zeroes of the polynomial

STEP 3: Determine the nature of the zeroes

STEP 4: Determine the y-intercept by evaluating $P(0)$

STEP 5: Sketch the curve

Question 11

Sketch the graph of $y = -5x(x-2)(x+4)$

STEP 1: Determine the general shape of the graph

STEP 2: Determine the zeroes of the polynomial

STEP 3: Determine the nature of the zeroes

STEP 4: Determine the *y*-intercept by evaluating $P(0)$.

STEP 5: Sketch the curve

Question 12

Graph the curve $y = x^3(x-1)(x-3)^2$

STEP 1: Determine the general shape of the graph

STEP 2: Determine the zeroes of the polynomial

STEP 3: Determine the nature of the zeroes

STEP 4: Determine the *y*-intercept by evaluating $P(0)$.

STEP 5: Sketch the curve

Question 13

Neatly sketch the graphs of the following polynomials, showing any important information

a) $y = (x + 4)(x - 1)(x - 3)$

b) Graph the curve $y = (x^2 - 2x - 15)(x^2 + x + 1)$ (You may assume the graph has a basic quartic shape)

POLYNOMIAL LONG DIVISION

DIVIDING INTEGERS

If we are asked to divide a number p by another number a we can write it as $p \div d = q \text{ "remainder" } \mathrm{r}$. Alternatively, we can write this as $p = a \cdot q + r$ where $0 \leq r \leq a - 1$

$$\begin{array}{r} 317 \\ 11\,\overline{)3489} \\ \underline{3300} \\ 189 \\ \underline{110} \\ 79 \\ \underline{77} \\ 2 \end{array}$$

An example is dividing 3489 by 11. Using the normal long division method we deduce that

$3489 = 317 \cdot 11 + 2$, i.e. the $q = 317$ and $r = 2$

DIVIDING POLYNOMIALS

Likewise if we are asked to divide a polynomial $P(x)$ by a polynomial $D(x)$, we can write this in the form $P(x) = D(x) \cdot Q(x) + R(x)$ where $Q(x)$is the quotient and $R(x)$is the remainder polynomial where $0 \leq \deg R(x) < \deg D(x) - 1$. A consequence of this is that $Q(x)$ and $R(x)$ are unique for every polynomial division.

Question 14 (Worked Example)

Use the division algorithm to divide $3x^3 + 4x^2 + 8x + 9$ by $x + 1$

Question 15

Using polynomial long division, divide the following, and express in the form $P(x) = A(x)Q(x) + R(x)$

a) $x^4 + 3x^2 - x + 6$ by $x - 3$

Talent Tip:

Talent Tip:

b) $x^3 + 2x^2 - 23x - 60$ by $x + 4$

c) $x^4 + 2x^3 - x^2 + 4x + 10$ by $x^2 - x + 2$

d) $3x^3 + 3x^2 + 6x - 1$ by $3x^2 + 1$

Talent Tip:

DIVISION THEOREM

When $P(x)$ is divided by $D(x)$, there remains unique polynomials $Q(x)$ and $R(x)$ such that -

- $P(x) = D(x) \times Q(x) + R(x)$
- $\deg R(x) < \deg D(x)$, or $R(x) = 0$

REMAINDER THEOREM

Question 16 – Proof of Remainder Theorem (Conceptual)

Suppose that the polynomial $P(x)$ is divided by $D(x) = (x - \alpha)$

a) Using the division theorem, write an expression for $P(x)$

b) Determine the degree of the remainder $R(x)$, and hence show that $P(x) = (x - a)Q(x) + r$, where r is a constant

c) Hence, show that $P(\alpha) = r$

Remainder Theorem

The remainder r of polynomial $P(x)$ upon division by a linear divisor $(x - a)$ is $P(a)$.

Question 17

Let $P(x) = x^3 - 3x^2 + 4x + 1$

a) Evaluate $P(1)$

b) Hence, find the remainder when $P(x)$ is divided by $(x - 1)$

Question 18

Find the remainder when the following polynomials are divided by the given divisor

a) $P(x) = x^3 - 3x^2 + 5$ is divided by $x - 4$

b) $P(x) = x^4 + x^3 + 2x^2 + 3x + 5$ is divided by x

c) $P(x) = x^3 + 2x^2 - 5x - 6$ is divided by $x + 3$

THE FACTOR THEOREM

Factor Theorem

$P(a) = 0$, then $(x - a)$ is a factor of $P(x)$ and vice versa

Question 19

a) Show that $x = 1, \pm 2$ are zeroes of $P(x) = x^3 - 4x - x^2 + 4$

b) Hence write $P(x)$ in factorise form

Question 20

Consider the polynomial $P(x) = 2x^3 - x^2 - 5x - 2$

a) Show that $x + 1$ is a factor

b) Using long division, factorise $P(x)$

Question 21

Given that $3x-1$ is a factor of $P(x)=3x^3-4x^2-17x+6$, find the other factors of $P(x)$