Foundations Maths.

Algebra 1

INTRODUCTION

So far, the majority of your mathematical journey has been concerned with real numbers, associated with values or quantities that are easy to imagine in space –things like distances, areas, times, lengths on a number line etc. They have a definite value and it is possible to know that value with certainty-it does not change.

Here we introduce a new facet of mathematics which is less associated with constant values, but rather with the relationship between numbers. As opposed to dealing with specific quantities (3 apples, 5m) we treat them in general (x apples, y metres) to establish broader relationships between ranges of values. This can be illuminating – by defining mathematical systems by the rules that relate them, it is possible to find solutions and solve problems we never could have before. It is far and away the most important tool in your mathematical toolbox.

If we really want to master the power of algebra, it is vital we understand the mathematical language behind it, and the rules that govern numbers in general. This will allow us to manipulate systems of numbers and relationships. These booklets aim to give you the basic skills before you launch into equations in the next booklet series.

Because these skills are so vital, it is crucial you are able to assess your ability in this field. The lists of problems given are by no means exhaustive – if you are still struggling see a tutor or mentor early and do more questions beyond the booklets. Conversely, if you find the questions here easy, try the challenge questions in the homework booklets.

DO NOT BE AFRAID TO MAKE MISTAKES – If you are not making mistakes, you're not learning. The mentors standing in front of you have probably spent as much time making mistakes whilst learning as they have getting things right, and that's why they are where they are there now. Get it wrong now, get it right in the exam, and always learn from your mistakes.

PRONUMERALS

A pronumeral is a letter or symbol which represents a value or number – usually because the exact value or number is unknown at the start of the problem. For example, I have x apples- here the x could mean 2, or 3 or 40 million – I do not know until I complete the rest of the question. The most commonly used pronumerals are x and y but you're allowed to use any Greek or Roman letter.

Because these pronumerals represent actual values, they ***must*** obey all the rules that govern values that you have seen before, though at first the way this works may seem obtuse. However a useful tip for checking how correct your algebra is, is to substitute in a random value and see if the algebraic expression still makes sense. We will provide a few examples of this technique in due course.

An important thing to realise is that we use algebra and pronumerals to illustrate relationships between numbers, or to represent other numbers. If we then are told the value of the pronumeral, we are then able to calculate other things. This is easiest to show with an example

Let a=5. Find the values of

i) a + 1

ii) a – 4

iii) 3a

iv) a ÷ 5

To solve these questions, we must substitute in our KNOWN value of a into the expressions given, to find the unknowns. Therefore

i) a + 1 = 5 + 1 = 6

ii) a – 4 = 5 – 4 = 1

iii) 3a = 3 x 5 = 15

iv) a ÷ 5 = 5 ÷ 5 = 1

Notice, all we have done is put a 5 in everywhere we used to see an a. This is called substitution.

Please note if we write two numbers or letter next to each other, this implies a multiplication i.e. $zy = z \times y$, or $3y = 3 \times y$

Now it's you're turn! Make sure you explicitly show your substitutions.

Question 1

a) Let $x = 4$. Find:

i. $x + 44 =$

ii. $x - 21 =$

iii. $4x =$

iv. $x \div 14 =$

v. $x^2 =$

vi. $(x+3)^2 - 14 =$

vii. $\sqrt{x + 12} =$

b) Let $x = 3$, Let $y = 2$. Find:

i. $x + y =$

ii. $xy =$

iii. $x \div y =$

iv. $x^3 y^2 =$

v. $x(x+y) =$

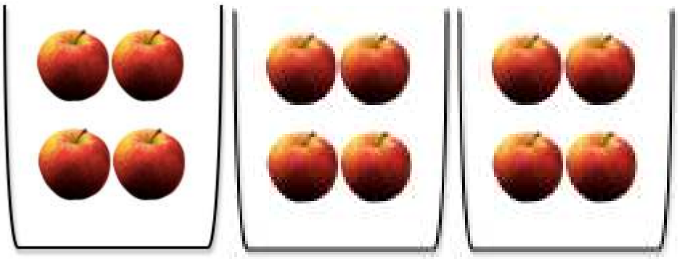
vi. $3\sqrt{x^2 + y^2 + 3} =$

vii. $\frac{y}{x} =$

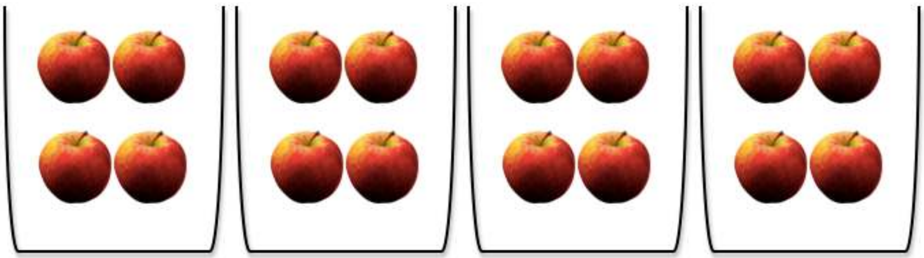
viii. $x! =$ (optional Question)

Collecting Like Terms

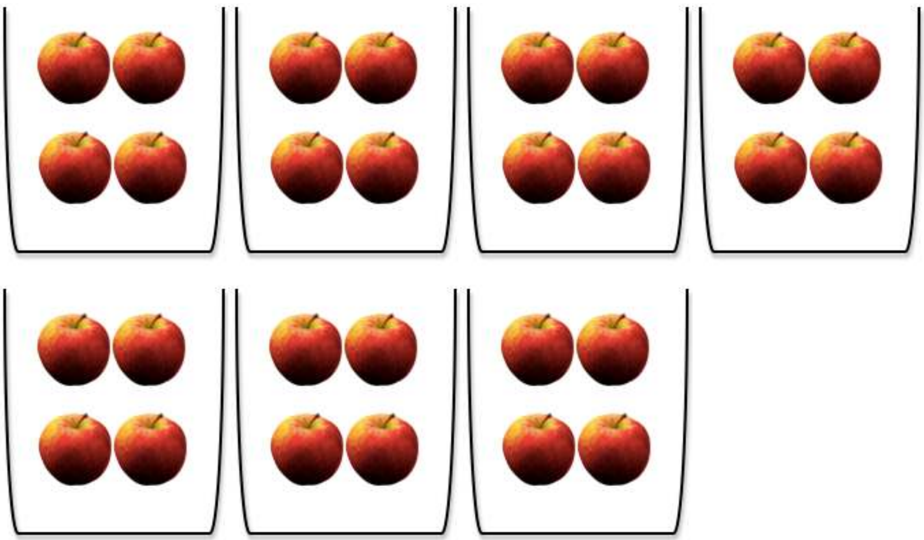
Imagine you have 3 buckets of apples with the same number of apples in each bucket.



You are then given 4 more buckets



How many do you now have? Evidently you now have 7 – hopefully you could do that without using your fingers!



Now if I told you that there were 4 apples in each bucket , and I asked you how many apples you had in total (i.e. across the 7 buckets), the answer to that is a simple multiplication – you have 7 lots (or buckets) of 4 apples, so you have $7 \times 4 = 28$ apples!

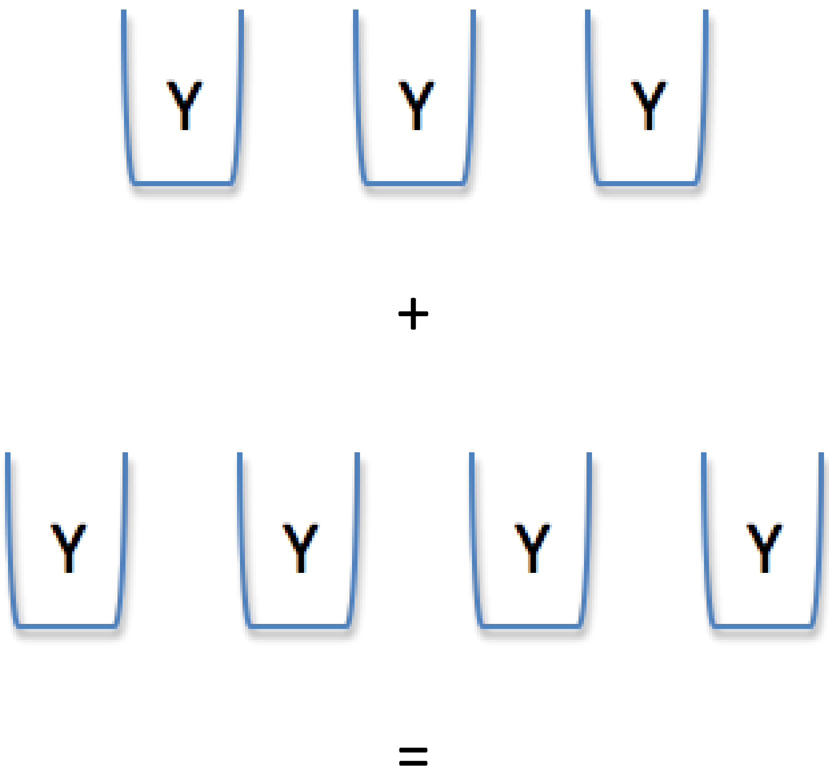
Notice you could also do this the long way. You started out with $3 \times 4 = 12$ apples, you were given $4 \times 4 = 16$ more apples, resulting in $12 + 16 = 28$ apples in total. You get the same answer!

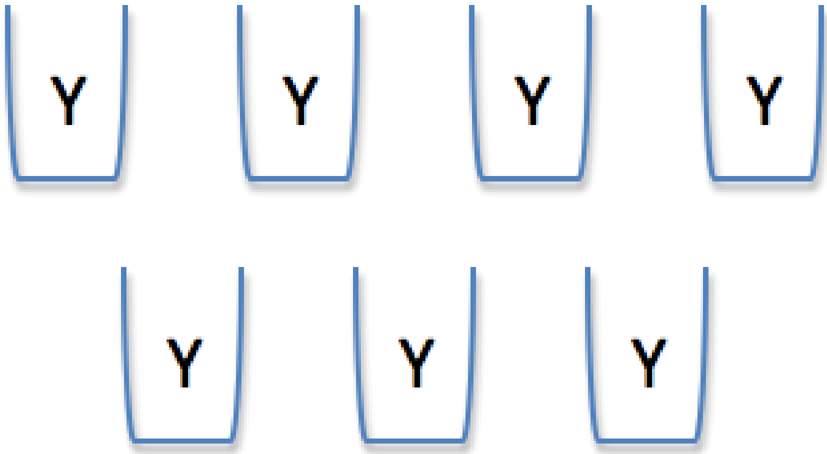
So now you remember how normal values work, imagine instead of having 4 apples per bucket, you now have Y apples per bucket, where Y could be ANY number of apples.

How many apples do you have at the beginning? Well you apply the same operation, you have 3 × Y apples, which we just write as 3Y apples

How many apples are you given? Again, you are given 4 lots of Y apples = 4 × Y = 4Y apples.

And how many apples do you have at the end. Well, you have 7 lots of Y apples = 7 × Y = 7Y apples.





Mathematically expressed, what we have just said is that

$$3Y + 4Y = 7Y$$

Simple right? But it illustrates a critical idea within algebra, Namely that WHEN TWO COEFFICIENTS ARE ASSOCIATED WITH EXACTLY THE SAME PRONUMERAL(S), THEY CAN BE ADDED OR SUBTRACTED AS YOU WOULD NORMAL NUMBERS - this is called collecting like terms.

Collecting Like Terms

If two coefficients are associated with **exactly the same pronumeral**, they can be added or subtracted as you would normal numbers

Some more examples – if you're confused, just think of the x's or y' or xyz's buckets of apples:

$$7x + 209x = 216x$$

$$215z - 213z = 2z$$

$$2xyz + 4xyz - xyz = 5xyz$$

$$7x^2y + 3x^2y = 10x^2y$$

$$x + x = 2x$$

$$4\theta^2\mu - 2\mu\theta^2 = 2\mu\theta^2 = 2\theta\mu^2$$

Note with the last example, that the order of the pronumerals does not matter, only their identity i.e $\theta^2 \mu = \mu\theta^2$, which means that the two can be collected and added as per normal. It doesn't matter which order you leave them in for your final answer unless the question specifies.

Common Errors

This may seem easy at first – after all, all you are doing is adding and subtracting numbers. However we guarantee that ALL of you will make one of the following mistakes at least once, so be aware of them (and if you don't, all the better for you!)

1) Mathematical errors: 15y – 7y is NOT equal to 9y (What is it equal to?!).

2) Collecting non-like terms: $15x^2y - 6y^2x$ CANNOT BE FURTHER SIMPLIFIED, as the pronumerals behind the coefficients are **DIFFERENT** (x^2y vs y^2x). This is a trick question. This is particularly important in more difficult questions where there are multiple different like terms to be collected e.g. $3x^3y + 4y^3x - 2xy^3 + 4yx^3 = 2xy^3 + 7x^3y$

3) Forgetting the coefficient of a pronumeral by itself (i.e. x, y, a) is 1

4) Running before they can walk – DIVISION AND MULTIPLICATION DO NOT WORK LIKE THIS (we're getting there, hold your horses!)

5) Forgetting addition and subtraction laws, particularly in the presence of minus signs.

$x + (3) = x+3$ $x - (+3) = x - 3$

$x + (-3) = x - 3$ $x - (-3) = x + 3$

Question 2

a) 3x + 2x =

b) 4y + 2y =

c) 6z – 2z =

d) $2z - 6z =$

e) $8y + 3x + 4x =$

f) $7y - 3x + 4y - 2x =$

g) $17x + 121x - 14x + x - 3y + 21y =$

h) $-3z + 4y + 15z =$

i) $-3g - (-5g) =$

j) $-5k - 3k - (-6k) =$

k) $3a - 4 - 8a =$

l) $x^2 + x^2 =$

m) $x - x^2 + 2x + 2 =$

n) $x^2y + xyz - 4x^2y + 12x^2y - yz =$

o) $6p^2q - 8pq^2 + 3p^2q - 12pq^2 =$

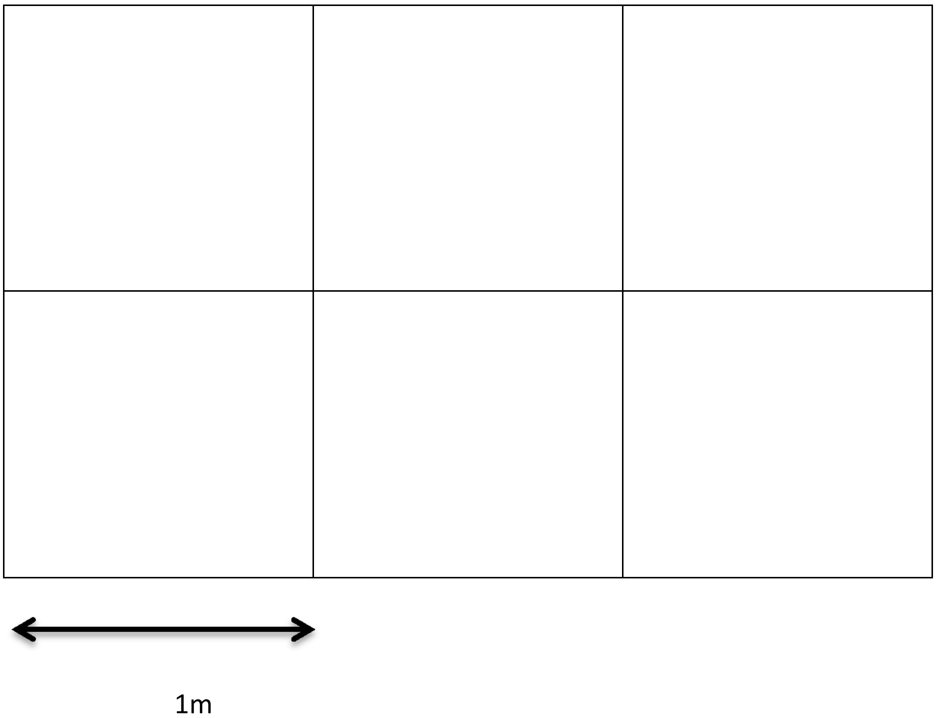
p) $-3s^2t + 9st + (-12st) =$

q) $7pq - (5pq - 7pq) + 19pq =$

r) $2p^2 - 7p - 2p - (-3p^2) =$

MULTIPLICATION

If you have a rectangle of 2m by 3m, we know the area is $2m \times 3m = 6m^2$



Notice that there are two things going on here

1) The numbers are multiplied together ($2 \times 3 = 6$)

2) The units (or pronumerals!) are multiplied together ($m \times m = m^2$)

This is also true of the general case. If pronumerals with coefficients are multiplied, the coefficients are multiplied and the pronumerals after them as well.

e.g. $3a \times 2b = 6ab$ $\quad$ $6y \times 2y = 12y^2$

Multiplying Pronumerals

When multiplying pronumerals, first multiply the coefficients, then multiply the pronumerals separately

Talent Tip:

Question 3

a) $90y \times 2x =$

b) $12\pi \times 3\sigma =$

c) $6\mu \times 4\theta =$

d) $5\rho \times 3\delta =$

Multiplying higher powers

We know from your other mathematics that the definition of $2^2=2\times2$ or that $3^4 = 3 \times 3 \times 3 \times 3$

This can also be applied to your pronumerals e.g. $y \times y = y^2$, $z \times z \times z \times z = z^4$

This is important when we do multiplication. Let's observe this in an example

$$z^2 \times z^3 = z \times z \times z \times z \times z = z^5$$

This lends itself to an important shortcut:

WHEN MULTIPLYING LIKE TERMS, THE INDICES OF THESE **LIKE TERMS** WILL **ADD**

e.g. $z^2 \times z^{604} = z^{604+2} = z^{606}$

Multiplying Higher Powers

When multiplying higher powers, the indices of **like terms** will add

Talent Tip:

Remember that the index of a z by itself is just 1. Therefore $z \times z^{105} = z^1 \times z^{105} = z^{1+105} = z^{106}$

Similarly this can actually be applied to like terms which are part of longer expressions

$3ab^2 \times 4b^3 = 3 \times a \times b \times b \times 4 \times b \times b \times b = 12ab^5$

Notice in the above example that the powers of b have added, the power of the "a" pronumeral has stayed the same (as there are no other a's in the expression), and the coefficients have multiplied.

Another example...

$7a^{214} b^{102} \times 6a^{135} b^{104} = 42a^{349} b^{206}$

Here, the coefficients have multiplied, the indices of the a's have added, and the indices of the b's have added.

Question 4

a) $a^3 \times a^7 =$

b) $a^{12} \times a^{14} =$

c) $3b^5 \times 2b^{17} =$

d) $12d^7 \times 6d^9 =$

e) $16d^{12}b^6 \times 4d^2b^3 =$

f) $3xy^2 \times 15y^4x \times 2y^8 =$

g) $xy \times x^2y \times xy^2 \times xyz =$

h) $(xyz)^2 \times (xyz)^4 =$

i) $2(ab^2) \times 4(ab^2)^5 =$

j) $b^x \times b^2 =$

k) $b^x \times b^y =$

l) $(a^3)^2$ (Note $(a^3)^2 = a^3 \times a^3$) =

m) $(3a^2)^3 =$

n) $(5a^4)^2 =$

PLEASE NOTE: If you replaced all the multiplication signs in the questions above with plus or minus signs, we CANNOT simplify most of the above expressions as they WOULD NOT BE LIKE TERMS. Multiplication (and division) are **not concerned with like terms**. This is a common error.

Powers of Powers

The last 3 questions of the last exercise are trying to illustrate a particular property of indices. You may have been doing these last three questions long hand-Turning these powers into multiplications. There is however a shortcut

$$(a^x b^y)^z = a^{xz} b^{yz}$$

In plain speak-you just multiply the indices. If there is a numerical coefficient, you can also evaluate.

Powers of powers

When applying a power to another power, the indices will **multiply**

Some examples...

$(3a^2b^3)^2 = 3^2 a^{2\times2} b^{3\times2} = 9a^4b^6$

$(4\mu\pi)^3 = 64\mu^3 \pi^3$ (Remember $\mu=\mu^1$)

Question 4

a) $(12\lambda\Psi)^2 =$

b) $(2xy^2z^5)^6 =$

c) $(abc^3)^4 \times 3bac =$

d) $(HEY)^5 \times (HI)^7 =$

e) $((2z^2y)^3)^2 =$

f) $(((2yz^2)^2)^2)^2 =$

g) $(abc)^2 \times 4(abc)^3 =$

h) $(abc)^2 \times (4abc)^3 =$

i) $(4xyz)^2 \times (2xyz)^3 =$

j) $(4xyz)^\theta \times (2xyz)^{2\theta} =$

Notice that by using algebra and the rules you have learnt you can simplify a lot of different questions-even questions involving things you may never have seen before! See for yourself!

Question 4

a) $4\tan^2\theta - 5\tan\theta - (16\tan\theta \text{ x } \tan\theta - 8\tan\theta)$

Let $\tan\theta = x$ (But remember to write your answer in terms of $\tan\theta$ at the end!)

b) $\text{cosec}^2x - 1 + 5\text{cosec}^2x - (7\text{cosec}x \times 4\text{cosec}^2x \div 14\text{cosec}x)$

Let $\text{cosec}x = y$

c) $\ln xy \times (\ln xy + 3\ln xy - 4\ln xy) - (\ln xy)^2$

Let $\ln xy = z$